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Lagrangian studies of plasma wave interactions I

T J M BOYD and J G TURNER

Department of Applied Mathematics, University of Wales, UCNW, Bangor, UK

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Abstract. A study of wave–wave interactions in plasmas is made using a lagrangian formulation developed by Low. Coupled mode equations are derived. The method offers distinct advantages over the conventional approach starting from the Vlasov–Maxwell equations. Two examples of the lagrangian method are considered: (i) the nonlinear interaction of transverse waves in a warm field-free plasma to produce plasma oscillations and (ii) the interaction of three electromagnetic waves in a cold magnetized plasma. Its application to waves in warm magnetized plasmas and to explosive instabilities is considered in part II.

1. Introduction

Studies of nonlinear wave interactions in plasmas are basic to an understanding of weak turbulence and are currently in the forefront of plasma research as laboratory studies are extended to nonlinear phenomena. At the outset it is important to understand that the label nonlinear is used to denote only *weak* departures from linear theory. Perturbations in the system are still expressed in terms of a set of linear modes but a weak interaction between modes is admitted. Thus in the linear combination of normal modes, coefficients are now slowly varying functions of time (or space) so that in the evolution of the system the state after some time is in general distinct from that predicted by linear theory. This theory—often referred to as weak turbulence theory—has as a requirement for its validity that the ratio of energy in the wave spectrum to the total energy in the plasma should be small. If this is not the case we are faced with strong plasma turbulence for which no adequate theory exists at present.

Nonlinear wave interactions in plasmas may be subdivided for convenience into wave–wave interactions and wave–particle interactions, of which Landau damping is a familiar example in the linear regime. We shall confine our attention to wave–wave interactions. Consider three plasma waves with frequencies ω_j satisfying dispersion relations $\omega_j = \omega_j(k_j)$, $j = 1, 2, 3$. Under conditions of resonance $\omega_3 = \omega_1 \pm \omega_2$, $\boldsymbol{k}_3 = \boldsymbol{k}_1 \pm \boldsymbol{k}_2$ wave coupling may play an important role in the plasma. In this interaction total energy and momentum in the wave system are conserved. Our separation of wave–wave interactions from nonlinear wave–particle interactions is of course arbitrary; in many turbulent plasmas they are in competition. However it is possible to devise experimental situations in which one or the other is dominant, at any rate within limited regimes of plasma parameters.

Studies of wave–wave interactions abound in the literature. The usual approach has followed the well beaten path of a perturbation treatment of the Vlasov–Maxwell set of equations. For the three wave interaction it leads—after rather tedious calculations—to a set of coupled mode equations

$$
\begin{aligned}
\mathrm{i}\omega_{1,2}^{-1}\left(\frac{\partial}{\partial t}+\boldsymbol{v}_{\mathrm{g}1,2}\cdot\nabla\right)\hat{A}_{1,2} &= \chi^*\hat{A}_{2,1}^*\hat{A}_3 \\
\mathrm{i}\omega_3^{-1}\left(\frac{\partial}{\partial t}+\boldsymbol{v}_{\mathrm{g}3}\cdot\nabla\right)\hat{A}_3 &= \chi\hat{A}_1\hat{A}_2
\end{aligned}
\tag{1}
$$

in which $\hat{A}_j$ is a wave amplitude, $\boldsymbol{v}_{\mathrm{g}j}$ is the group velocity of wave j, and χ is the coupling coefficient, asterisks denoting complex conjugates.

From the coupled mode equations one finds

$$
\omega_1^{-1}\mathrm{D}_1\epsilon_1 = \omega_2^{-1}\mathrm{D}_2\epsilon_2 = -\omega_3^{-1}\mathrm{D}_3\epsilon_3 = \mathrm{i}\chi\hat{A}_1\hat{A}_2\hat{A}_3^* + \text{complex conjugate} \tag{2}
$$

with $\mathrm{D}_j = \partial/\partial t+\boldsymbol{v}_{\mathrm{g}j}\cdot\nabla$, $\epsilon_j = \hat{A}_j\hat{A}_j^*$. These are the Manley–Rowe relations expressing the conservation of wave action (Sturrock 1960).

There is however an alternative to such a procedure based on a study of the Lagrangian for the plasma, first formulated by Low (1958). Low applied the lagrangian formalism to study plasma oscillations and hydromagnetic waves in linear theory. Apart from some work by Suramlishvili (1964, 1965 and 1967) there appears to have been little realization of the benefits of a lagrangian approach in studying nonlinear problems in plasma physics until recently (Galloway and Crawford 1970).

However a lagrangian approach has been examined independently in fluid mechanics by Whitham (1965) who introduced the idea of an averaged Lagrangian in a general approach to linear and nonlinear dispersive waves. Bretherton and Garrett (1968) used the concept of an averaged Lagrangian to demonstrate the conservation of wave action for a wide class of conservative systems in fluid dynamics. Dougherty (1970) has generalized their work in a relativistic treatment and has extended it to include the nonlinear interaction between waves. In particular he considers three-wave interactions and derives a conservation law which is the general form taken by the Manley–Rowe relations (cf (2)).

A specific application of the lagrangian approach has been made by Simmons (1969) to derive coupled-mode equations describing weak resonant wave interactions of capillary-gravity waves in an inviscid fluid. In this paper and its sequel we wish to consider some wave–wave interactions of interest in plasma physics from the lagrangian standpoint.

The procedure adopted in the lagrangian approach is the following. The plasma is described in terms of Low's Lagrangian which is then developed in an expansion scheme. This describes in its various orders the equilibrium state of the plasma, the linear wave spectrum, three-wave coupling processes, four-wave coupling etc. The familiar Bogoliubov–Krylov multiple scale expansion is then used on the independent variables. In the case of two spatial and two time scales one (fast, $\boldsymbol{x}$, t) is characteristic of the wavenumbers and frequencies of the normal modes of the system while the other (slow, $\epsilon\boldsymbol{x}$, ϵt) characterizes the space–time variation of the amplitudes of the coupled modes. This procedure leads to action integrals for the unperturbed plasma on the one hand and for the waves and their interactions on the other. Use of a space–time averaged Lagrangian together with corresponding averaged energy densities and fluxes then gives general expressions for the coupled mode equations together with the coupling coefficients.

The averaging procedure is discussed in § 2 where the general theory is presented. The application of the method to two specific examples of three-wave interactions is considered in § 3, namely the conversion of electromagnetic waves into longitudinal plasma oscillations in a warm, field-free plasma and the interaction of three transverse waves in a cold magnetized plasma. The lagrangian approach to wave–wave interactions in plasmas is discussed in § 4.

2. Theory

Consider a plasma described by the Lagrangian

$$L = \int\int \mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{v}\,\mathscr{L}\left(q^i, \frac{\partial q^i}{\partial t}, \frac{\partial q^i}{\partial \boldsymbol{x}}, \frac{\partial q^i}{\partial \boldsymbol{v}}, \boldsymbol{x}, \boldsymbol{v}\right) \tag{3}$$

where $\{q^i(\boldsymbol{x}, \boldsymbol{v}, t)\}, i = 1, 2, \dots p$, is the set of generalized variables and $\mathscr{L}$ is the lagrangian density. In § 1 we limited the present discussion to wave–wave interactions only; in particular we shall be interested in those in which total wave energy and momentum are conserved, that is, $\mathscr{L}$ shows no explicit time dependence. In this problem, the generalized variables are the position $\boldsymbol{r}$ of the particle together with the electrostatic and electromagnetic potentials ϕ and $\boldsymbol{A}$ respectively. For brevity, we shall denote the functional dependence of $\mathscr{L}$ given in (3) by $\mathscr{L}(q^i, \boldsymbol{x}, \boldsymbol{v})$. The equations of motion for the plasma follow from Hamilton's principle, that is $S = \iiint \mathscr{L}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}t$ is stationary and are given by the Euler–Lagrange equations

$$\frac{\delta}{\delta t}\left(\frac{\partial \mathscr{L}}{\partial(\partial_t q^i)}\right) + \frac{\delta}{\delta \boldsymbol{x}}\cdot\left(\frac{\partial \mathscr{L}}{\partial(\partial_{\boldsymbol{x}} q^i)}\right) + \frac{\delta}{\delta \boldsymbol{v}}\cdot\left(\frac{\partial \mathscr{L}}{\partial(\partial_{\boldsymbol{v}} q^i)}\right) - \frac{\partial \mathscr{L}}{\partial q^i} = 0 \tag{4}$$

($i = 1, 2, \dots p$), where $\partial_t \equiv \partial/\partial t$ and similarly for $\partial_{\boldsymbol{x}}, \partial_{\boldsymbol{v}}$. The symbol δ represents partial differentiation in the sense that $\boldsymbol{x}, \boldsymbol{v}, t$ are independent variables, but at the same time, the dependence of $\mathscr{L}$ on $\boldsymbol{x}, \boldsymbol{v}, t$ through $\partial_{\boldsymbol{x}} q^i, \partial_{\boldsymbol{v}} q^i, \partial_t q^i$ is also taken into account. Define a generalized energy density

$$\mathscr{H}(q^i, \boldsymbol{x}, \boldsymbol{v}) = \sum_i \frac{\partial \mathscr{L}}{\partial(\partial_t q^i)} - \mathscr{L} \tag{5}$$

and corresponding energy fluxes in phase-space $\mathscr{P}^{\mathrm{s}}, \mathscr{P}^{\mathrm{v}}$

$$\mathscr{P}^{\mathrm{s}}(q^i, \boldsymbol{x}, \boldsymbol{v}) = \sum_i \frac{\partial \mathscr{L}}{\partial(\partial_{\boldsymbol{x}} q^i)} \partial_t q^i \tag{6}$$

$$\mathscr{P}^{\mathrm{v}}(q^i, \boldsymbol{x}, \boldsymbol{v}) = \sum_i \frac{\partial \mathscr{L}}{\partial(\partial_{\boldsymbol{v}} q^i)} \partial_t q^i \tag{7}$$

where $\mathscr{P}^{\mathrm{s}}, \mathscr{P}^{\mathrm{v}}$ denote space and velocity components respectively. Write

$$q^i = q^i_0 + \epsilon \eta^i \tag{8}$$

where the q^i_0 represent the equilibrium state and the η^i denote the total perturbation of this state due to the waves and their interactions.

The small parameter ϵ denotes the ratio of the time (length) scales, contained in the prescription of Krylov and Bogoliubov (1949); the slow scale characterizes the variation in time (space) of the amplitudes of the coupled modes while the fast is associated with the natural frequencies (wavenumbers) of the modes. It follows from Hamilton's principle under the change of variable (8) that the $\{\eta^i(\boldsymbol{x}, \boldsymbol{v}, t)\}$ may now be regarded as the independent generalized variables with the $\{q_0^i(\boldsymbol{x}, \boldsymbol{v})\}$ regarded as known. The equations of motion of the $\{\eta^i\}$ are then determined from (4) with q^i replaced by η^i, and in direct analogy with (5)–(7), we may define the energy density $\mathscr{H}(\eta^i, \boldsymbol{x}, \boldsymbol{v})$ and the flux vectors $\mathscr{P}^{\mathrm{s}}(\eta^i, \boldsymbol{x}, \boldsymbol{v})$, $\mathscr{P}^{\mathrm{v}}(\eta^i, \boldsymbol{x}, \boldsymbol{v})$. On substituting (8) together with its derivatives in (3), a formal expansion of $\mathscr{L}$ may be made in powers of the perturbation

$$\mathscr{L} = \mathscr{L}_0 + \epsilon\mathscr{L}_1 + \epsilon^2\mathscr{L}_2 + \epsilon^3\mathscr{L}_3 + \ldots \tag{9}$$

where $\mathscr{L}_n$ is a homogeneous expression of degree n in the η^i, $\partial_t\eta^i$, $\partial_{\boldsymbol{x}}\eta^i$, $\partial_{\boldsymbol{v}}\eta^i$. Corresponding expansions for $\mathscr{H}$, $\mathscr{P}^{\mathrm{s}}$ and $\mathscr{P}^{\mathrm{v}}$ may also be written in powers of the perturbation η^i. Since $\mathscr{L}_0$ is a function of the equilibrium quantities q_0^i only, it describes the equilibrium state. $\mathscr{L}_1$ is of first order in the $\{\eta^i\}$ and their derivatives and so has no effect on the equation of motion for $\{\eta^i\}$. In fact Low has shown that L_1 vanishes in general as is apparent in § 2.1 from the fact that $f_0(\boldsymbol{x}, \boldsymbol{v})$ is a solution of the time independent Vlasov equation. The variation of $\mathscr{L}_2$ with respect to the η^i yields the linear Vlasov–Maxwell equations describing wave propagation in the plasma. The physics of wave interactions is contained in third and higher order terms; in order to describe three-wave interactions, the perturbation series is truncated after $\mathscr{L}_3$. In the linear regime for some parameter U_n of the nth wave

$$U_n = \mathrm{Re}\{\hat{U}_n \exp \mathrm{i}(\boldsymbol{k}_n \cdot \boldsymbol{x} - \omega_n t)\} \tag{10}$$

where $\hat{U}_n$ is constant in time and space. To describe wave coupling we suppose that solutions of the nonlinear equation have the form (10) with $\hat{U}_n$ now regarded as a slowly varying function of $\boldsymbol{x}$ and t.

The perturbation η^i is now separated into its individual wave components, so that for the three-wave interactions of interest

$$\eta^i = \eta_1^i + \eta_2^i + \eta_3^i \tag{11}$$

with each η_n^i ($n = 1, 2, 3$) given by (10). One now defines a Lagrangian *for each wave*

$$\mathscr{L}(n) = \mathscr{L}\left(\eta_n^i(\boldsymbol{x}, \boldsymbol{v}, t) + \left(q_0^i(\boldsymbol{x}, \boldsymbol{v}) + \sum_{m \neq n} \eta_m^i(\boldsymbol{x}, \boldsymbol{v}, t)\right), \boldsymbol{x}, \boldsymbol{v}\right) \tag{12}$$

which is now time dependent, that is

$$\mathscr{L}(n) \equiv \mathscr{L}(\eta_n^i(\boldsymbol{x}, \boldsymbol{v}, t), \boldsymbol{x}, \boldsymbol{v}, t).$$

In these individual wave Lagrangians we treat the $\{\eta_n^i(\boldsymbol{x}, \boldsymbol{v}, t)\}$ as the set of generalized variables, whilst all other wave variables $\eta_m^i(\boldsymbol{x}, \boldsymbol{v}, t)$ $m \neq n$, $q_0^i(\boldsymbol{x}, \boldsymbol{v})$ are regarded as *explicit* functions of $\boldsymbol{x}, \boldsymbol{v}, t$ which are known. Thus $\partial/\partial t$ acting on $\mathscr{L}(n)$ will operate only on the $\{\eta^i_{m \neq n}\}$.

We now define the corresponding time-dependent energy density and flux vectors for each wave in analogy with (5)–(7)

$$\mathscr{H}(n) = \sum_i \frac{\partial\mathscr{L}}{\partial(\partial_t\eta_n^i)}\partial_t\eta_n^i - \mathscr{L}(n) \tag{13}$$

$$\mathscr{P}^{\mathrm{s}}(n) = \sum_i \frac{\partial\mathscr{L}}{\partial(\partial_{\boldsymbol{x}}\eta_n^i)}\partial_t\eta_n^i \tag{14}$$

$$\mathscr{P}^{\mathrm{v}}(n) = \sum_i \frac{\partial\mathscr{L}}{\partial(\partial_{\boldsymbol{v}}\eta_n^i)}\partial_t\eta_n^i. \tag{15}$$

The coupled-mode equations then follow by considering the time rate of change of the time-dependent energy per unit volume

$$H(n) = \int \mathscr{H}(n)\,\mathrm{d}\boldsymbol{v} \tag{16}$$

with $\mathscr{H}(n)$ given by (13). The time rate of change in this case is

$$\frac{\delta}{\delta t} = \sum_i \left((\partial_t \eta_n^i)\frac{\partial}{\partial \eta_n^i} + \partial_t(\partial_t \eta_n^i)\frac{\partial}{\partial(\partial_t \eta_n^i)} + \partial_t(\partial_{\boldsymbol{x}} \eta_n^i)\cdot\frac{\partial}{\partial(\partial_{\boldsymbol{x}} \eta_n^i)} + \partial_t(\partial_{\boldsymbol{v}} \eta_n^i)\cdot\frac{\partial}{\partial(\partial_{\boldsymbol{v}} \eta_n^i)}\right) + \frac{\partial}{\partial t}.$$

Using (13) and eliminating $(\delta/\delta t)(\partial\mathscr{L}/\partial(\partial_t\eta_n^i))$ via the Euler–Lagrange equations leads to

$$\int \frac{\delta\mathscr{H}(n)}{\delta t}\,\mathrm{d}\boldsymbol{v} + \int\left(\frac{\delta}{\delta\boldsymbol{x}}\cdot\mathscr{P}^{\mathrm{s}}(n) + \frac{\delta}{\delta\boldsymbol{v}}\cdot\mathscr{P}^{\mathrm{v}}(n)\right)\mathrm{d}\boldsymbol{v} = -\int\frac{\partial\mathscr{L}(n)}{\partial t}\,\mathrm{d}\boldsymbol{v}. \tag{17}$$

We now make use of multiple time and length scales in order to describe weak nonlinear coupling. The amplitudes of the interacting waves are assumed to be functions of space and time which vary slowly compared with the variation of equilibrium quantities so that one may introduce a space–time averaging procedure. This averaging, denoted by a bar, is one taken over intervals of space and periods of time which are long compared with periods of oscillation of the uncoupled linear waves, k_n^{-1}, ω_n^{-1}, but short compared with intervals and periods over which the amplitudes of the interacting waves vary appreciably. The effect of the averaging is to separate synchronous terms, that is, those which have no net phase dependence, from those with nonzero phase which vanish. The space–time averaged quantities are then all slowly varying functions of $\boldsymbol{x}$ and t. It is the application of this space–time averaging to (17) which leads directly to the coupled-mode equations and coupling coefficient, that is

$$\int \overline{\frac{\delta\mathscr{H}(n)}{\delta t}}\,\mathrm{d}\boldsymbol{v} + \int\overline{\frac{\delta}{\delta\boldsymbol{x}}\cdot\mathscr{P}^{\mathrm{s}}(n)}\,\mathrm{d}\boldsymbol{v} = -\int\overline{\frac{\partial\mathscr{L}(n)}{\partial t}}\,\mathrm{d}\boldsymbol{v}. \tag{18}$$

The third term in (17) is not displayed since $(\delta/\delta\boldsymbol{v})\cdot\boldsymbol{A}(\boldsymbol{x},t) \equiv 0$. The coupled-mode equations then follow from (18) where we write $\mathscr{L}(n) = \mathscr{L}_2(n)+\mathscr{L}_3(n)$. We indicate how the various terms in (18) are evaluated by giving a detailed account of the calculation for $\int\overline{(\partial\mathscr{L}(n)/\partial t)}\,\mathrm{d}\boldsymbol{v}$. Consider (18) for wave 1; since $\mathscr{L}_2$ is of the form

$$\mathscr{L}_2 = \sum_{j,l} a_{jl}\eta^j\eta^l$$

from (10)–(12), we have

$$\begin{aligned}\int\frac{\partial}{\partial t}\mathscr{L}_2(1)\,\mathrm{d}\boldsymbol{v} = \frac{1}{4}\frac{\partial}{\partial t}\int\sum_{j,l} a_{jl}[\{(\hat{\eta}_1^j\,\mathrm{e}^{\mathrm{i}\alpha_1}+\hat{\eta}_1^{j*}\,\mathrm{e}^{-\mathrm{i}\alpha_1})+(\hat{\eta}_2^j(\boldsymbol{x},\boldsymbol{v},t)\,\mathrm{e}^{\mathrm{i}\alpha_2}\\
+\hat{\eta}_2^{j*}(\boldsymbol{x},\boldsymbol{v},t)\,\mathrm{e}^{-\mathrm{i}\alpha_2})+(\hat{\eta}_3^j(\boldsymbol{x},\boldsymbol{v},t)\,\mathrm{e}^{\mathrm{i}\alpha_3}+\hat{\eta}_3^{j*}(\boldsymbol{x},\boldsymbol{v},t)\,\mathrm{e}^{-\mathrm{i}\alpha_3})\}\\
\times\{(\hat{\eta}_1^l\,\mathrm{e}^{\mathrm{i}\alpha_1}+\hat{\eta}_1^{l*}\,\mathrm{e}^{-\mathrm{i}\alpha_1})+(\hat{\eta}_2^l(\boldsymbol{x},\boldsymbol{v},t)\,\mathrm{e}^{\mathrm{i}\alpha_2}\\
+\hat{\eta}_2^{l*}(\boldsymbol{x},\boldsymbol{v},t)\,\mathrm{e}^{-\mathrm{i}\alpha_2})+(\hat{\eta}_3^l(\boldsymbol{x},\boldsymbol{v},t)\,\mathrm{e}^{\mathrm{i}\alpha_3}\\
+\hat{\eta}_3^{l*}(\boldsymbol{x},\boldsymbol{v},t)\,\mathrm{e}^{-\mathrm{i}\alpha_3})\}]\,\mathrm{d}\boldsymbol{v}\end{aligned} \tag{19}$$

where $\alpha_n = \boldsymbol{k}_n \cdot \boldsymbol{x} - \omega_n t$. Performing the space–time average gives

$$\int \overline{\frac{\partial}{\partial t}\mathscr{L}_2(1)}\,\mathrm{d}\boldsymbol{v} = \frac{\partial}{\partial t}\frac{1}{4}\int \sum_{j,l} a_{jl}\{(\hat{\eta}_1^j \hat{\eta}_1^{l*} + \mathrm{cc}) + (\hat{\eta}_2^j(\boldsymbol{x}, \boldsymbol{v}, t)\hat{\eta}_2^{l*}(\boldsymbol{x}, \boldsymbol{v}, t) + \mathrm{cc}) + (\hat{\eta}_3^j(\boldsymbol{x}, \boldsymbol{v}, t)\hat{\eta}_3^{l*}(\boldsymbol{x}, \boldsymbol{v}, t) + \mathrm{cc})\}\,\mathrm{d}\boldsymbol{v}$$

where cc denotes complex conjugates. Remembering that $\partial/\partial t$ operates only on waves 2 and 3

$$\int \overline{\frac{\partial}{\partial t}\mathscr{L}_2(1)}\,\mathrm{d}\boldsymbol{v} = \int \left(\frac{\partial}{\partial t}\overline{\mathscr{L}_2^2} + \frac{\partial}{\partial t}\overline{\mathscr{L}_2^3}\right)\mathrm{d}\boldsymbol{v}$$

where the generic wave Lagrangian $\mathscr{L}_2^n \equiv \mathscr{L}_2(\eta_n^i, \boldsymbol{x}, \boldsymbol{v})$. Similarly

$$\mathscr{L}_3 = \sum_{j,l,m} a_{jlm}\eta^j\eta^l\eta^m$$

from which we find on space–time averaging

$$\int \overline{\frac{\partial}{\partial t}\mathscr{L}_3(1)}\,\mathrm{d}\boldsymbol{v} = \int \left(\frac{\partial}{\partial t}\frac{1}{8}\sum_{j,l,m} a_{jlm}\{(\hat{\eta}_1^j\,\mathrm{e}^{\mathrm{i}\alpha_1})\hat{\eta}_2^l(\boldsymbol{x}, \boldsymbol{v}, t)\,\mathrm{e}^{\mathrm{i}\alpha_2}\hat{\eta}_3^{m*}(\boldsymbol{x}, \boldsymbol{v}, t)\,\mathrm{e}^{-\mathrm{i}\alpha_1} + \mathrm{cc} + \hat{\eta}_2^j(\boldsymbol{x}, \boldsymbol{v}, t)\,\mathrm{e}^{\mathrm{i}\alpha_2}(\hat{\eta}_1^l\,\mathrm{e}^{\mathrm{i}\alpha_1})\hat{\eta}_3^{m*}(\boldsymbol{x}, \boldsymbol{v}, t)\,\mathrm{e}^{-\mathrm{i}\alpha_3} + \mathrm{cc} + \hat{\eta}_3^{j*}(\boldsymbol{x}, \boldsymbol{v}, t)\,\mathrm{e}^{-\mathrm{i}\alpha_3}\hat{\eta}_2^l(\boldsymbol{x}, \boldsymbol{v}, t)\,\mathrm{e}^{\mathrm{i}\alpha_2}(\hat{\eta}_1^m\,\mathrm{e}^{\mathrm{i}\alpha_1}) + \mathrm{cc}\}\right)\mathrm{d}\boldsymbol{v}$$

where the term in wave 1 has been bracketed to show that it is *not* to be differentiated when the differentiation with respect to time is carried out. Then

$$\int \overline{\frac{\partial}{\partial t}\mathscr{L}_3(1)}\,\mathrm{d}\boldsymbol{v} = \frac{\mathrm{i}}{8}(\omega_3 - \omega_2)\int\left(\sum_{j,l,m} a_{jlm}(\hat{\eta}_1^j\hat{\eta}_2^l\hat{\eta}_3^{m*} + \hat{\eta}_1^l\hat{\eta}_2^j\hat{\eta}_3^{m*} + \hat{\eta}_1^m\hat{\eta}_2^l\hat{\eta}_3^{j*}) + \mathrm{cc}\right)\mathrm{d}\boldsymbol{v}. \tag{20}$$

From the synchronism condition $\omega_3 - \omega_2 = \omega_1$, we can write (20) as

$$\int \overline{\frac{\partial}{\partial t}\mathscr{L}_3(1)}\,\mathrm{d}\boldsymbol{v} = \omega_1\epsilon_{\mathrm{wc}} \tag{21}$$

where ϵ_{wc} denotes the wave coupling energy. A similar analysis shows that to lowest order in the nonlinearity

$$\int \overline{\frac{\delta\mathscr{H}(1)}{\delta t}}\,\mathrm{d}\boldsymbol{v} = \int\left(\frac{\delta}{\delta t}\overline{\mathscr{H}_2^1} - \frac{\partial}{\partial t}\overline{\mathscr{L}_2^2} - \frac{\partial}{\partial t}\overline{\mathscr{L}_2^3}\right)\mathrm{d}\boldsymbol{v}$$
$$= \frac{\delta}{\delta t}\epsilon_1 - \int\frac{\partial}{\partial t}\left(\overline{\mathscr{L}_2^2} + \overline{\mathscr{L}_2^3}\right)\mathrm{d}\boldsymbol{v}$$

where $\epsilon_n \equiv \int \overline{\mathscr{H}_2(\eta_n^i, \boldsymbol{x}, \boldsymbol{v})}\,\mathrm{d}\boldsymbol{v}$. Finally

$$\int \overline{\frac{\delta}{\delta \boldsymbol{x}}\cdot\mathscr{P}^{\mathrm{s}}(1)}\,\mathrm{d}\boldsymbol{v} = \frac{\delta}{\delta \boldsymbol{x}}\cdot\boldsymbol{P}_1^{\mathrm{s}}$$

where $\boldsymbol{P}_n^{\mathrm{s}} \equiv \int \overline{\mathscr{P}_2^{\mathrm{s}}(\eta_n^i, \boldsymbol{x}, \boldsymbol{v})}\, \mathrm{d}\boldsymbol{v}$. Performing the same analysis on (18) with waves 2 and 3 results in the set of equations

$$c_n\left(\frac{\delta\epsilon_n}{\delta t}+\frac{\delta}{\delta \boldsymbol{x}}\cdot \boldsymbol{P}_n^{\mathrm{s}}\right) = -\omega_n\epsilon_{\mathrm{wc}} \qquad (n = 1, 2, 3) \tag{22}$$

where $c_1 = c_2 = 1$ and $c_3 = -1$.

Stix (1962) and Allis *et al* (1963) have shown that the group velocity $\boldsymbol{v}_{\mathrm{g}}$ is equal to the velocity of transport of energy, where this quantity is defined as the ratio of the total space–time averaged energy flux to the total space–time averaged density, that is

$$\boldsymbol{v}_{\mathrm{g}n} = \frac{\boldsymbol{P}_n^{\mathrm{s}}}{\epsilon_n}.$$

Here ϵ_n denotes the sum of the magnetic and electrostatic energy densities together with that part of the charged particle kinetic energy associated with the coherent wave motion, while $\boldsymbol{P}_n^{\mathrm{s}}$ denotes the sum of electromagnetic energy flux and that deriving from the coherent wave motion.

This identification enables (22) to be written

$$c_n\left\{\left(\frac{\delta}{\delta t}+\boldsymbol{v}_{\mathrm{g}n}\cdot\frac{\delta}{\delta \boldsymbol{x}}\right)\epsilon_n+\epsilon_n\left(\frac{\delta}{\delta \boldsymbol{x}}\cdot \boldsymbol{v}_{\mathrm{g}n}\right)\right\} = -\omega_n\epsilon_{\mathrm{wc}} \tag{23}$$

and since $\epsilon_n \equiv \epsilon_n(\boldsymbol{x}, t)$, then

$$\frac{\delta}{\delta t}\epsilon_n \to \frac{\partial\epsilon_n}{\partial t} \qquad \text{and} \qquad \frac{\delta}{\delta \boldsymbol{x}}\epsilon_n \to \frac{\partial\epsilon_n}{\partial \boldsymbol{x}}.$$

Using $\mathrm{D}_n \equiv (\partial/\partial t)+\boldsymbol{v}_{\mathrm{g}n}\cdot\nabla$, (23) may be written

$$c_n\omega_n^{-1}\{\mathrm{D}_n\epsilon_n+\epsilon_n(\nabla\cdot\boldsymbol{v}_{\mathrm{g}n})\} = -\epsilon_{\mathrm{wc}}. \tag{24†}$$

For the interactions under discussion, the group velocity is spatially independent so that (24) becomes

$$c_n\omega_n^{-1}\mathrm{D}_n\epsilon_n = -\epsilon_{\mathrm{wc}}. \tag{25}$$

We may identify ϵ_n/ω_n with the *action* of wave n and (25) can be written

$$\omega_1^{-1}\mathrm{D}_1\epsilon_1 = \omega_2^{-1}\mathrm{D}_2\epsilon_2 = -\omega_3^{-1}\mathrm{D}_3\epsilon_3.$$

These are the *action-transfer relations* first discussed by Sturrock (1960) in the context of plasma physics. It may be seen that the rates of transfer of action between waves participating in an interaction are in the ratio $1:\pm 1$. Further, writing (25) in the form

$$\omega_1^{-1}\mathrm{D}_1\epsilon_1+\omega_3^{-1}\mathrm{D}_3\epsilon_3 = 0 = \omega_2^{-1}\mathrm{D}_2\epsilon_2+\omega_3^{-1}\mathrm{D}_3\epsilon_3 \tag{26}$$

and defining $\omega_{rs} = r\omega_1+s\omega_2$ so that $\omega_{10} = \omega_1, \omega_{01} = \omega_2, \omega_{11} = \omega_3$ with similar definitions for D_{rs} and ϵ_{rs}, (26) becomes

$$\sum_{r,s} r\frac{\mathrm{D}_{rs}\epsilon_{rs}}{r\omega_1+s\omega_2} = 0 = \sum_{r,s} s\frac{\mathrm{D}_{rs}\epsilon_{rs}}{r\omega_1+s\omega_2}$$

† When wave coupling is ignored, (24) is a particular case of the general result $c_n\{\mathrm{D}_n(\epsilon_n/\omega_n)+(\nabla\cdot\boldsymbol{v}_{\mathrm{g}n})\epsilon_n/\omega_n\} = 0$ for constant ω_n. This last equation is the principal result of a paper by Bretherton and Garrett (1968) and states that in a time dependent, nonuniformly moving medium, the total wave action ϵ_n/ω_n is conserved along a ray. A ray is the path traced in space–time by an observer moving with the local value of $\boldsymbol{v}_{\mathrm{g}n}$.

which are the Manley–Rowe relations originally derived from the consideration of energy transfer within nonlinear elements of electrical networks (Penfield 1960).

Finally, from $\mathscr{L}_2$ we derive the linear equations of motion which enable us to express the η_n^i in terms of one wave parameter, say a_n^i, that is $\eta_n^i = \alpha^i a_n^i$. With this substitution, we find

$$\epsilon_n = \Gamma_n \hat{a}_n \hat{a}_n^* \qquad \boldsymbol{P}_n^s = \Lambda_n \hat{a}_n \hat{a}_n^* \qquad \epsilon_{wc} = \mathrm{i}\Gamma_{wc}\hat{a}_1\hat{a}_2\hat{a}_3^* + \mathrm{cc}. \tag{27a, b, c}$$

Then for wave 1, (24) becomes

$$\frac{\partial}{\partial t}\Gamma_1 \hat{a}_1 \hat{a}_1^* + (\boldsymbol{v}_{g1} \,.\, \nabla)\Gamma_1 \hat{a}_1 \hat{a}_1^* = -\mathrm{i}\omega_1 \Gamma_{wc}\hat{a}_1\hat{a}_2\hat{a}_3^* + \mathrm{cc}.$$

Taking $\Gamma_n > 0$ and defining $\hat{A}_n = \Gamma_n^{1/2}\hat{a}_n$, this equation becomes

$$\hat{A}_1^* \mathrm{D}_1 \hat{A}_1 + \mathrm{cc} = \mathrm{i}\omega_1 \frac{\Gamma_{wc}^*}{(\Gamma_1\Gamma_2\Gamma_3)^{1/2}}\hat{A}_1^*\hat{A}_2^*\hat{A}_3 + \mathrm{cc}. \tag{28}$$

Comparison of (28) with (1) shows that we may drop the complex conjugate from each side of (28) and identify the wave coupling coefficient

$$\chi = -\frac{\Gamma_{wc}}{(\Gamma_1\Gamma_2\Gamma_3)^{1/2}}.$$

Equation (28) then becomes identical with (1), giving the coupled-mode equations.

2.1. Lagrangian for a warm plasma

We conclude this section with a statement of the main results from the paper of Low concerning the warm plasma Lagrangian. The lagrangian density for a hot electron plasma in a magnetic field $\boldsymbol{B}_0$ is given by

$$\mathscr{L} = f_0(\boldsymbol{x}, \boldsymbol{v})\left\{\frac{m}{2}(\mathrm{D}_\Omega \boldsymbol{r})^2 + e\left(\phi - \frac{\boldsymbol{v} \,.\, \boldsymbol{A}}{c}\right)\right\} + \frac{\chi(\boldsymbol{v})}{8\pi}\left\{\left(\nabla\phi + \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}\right)^2 - (\nabla \times \boldsymbol{A})^2\right\} \tag{29}$$

where $-e(e > 0)$ and m are the charge and mass of an electron, f_0 is the equilibrium distribution function and $\chi(\boldsymbol{v})$ is an arbitrary function of velocity such that $\int \chi(\boldsymbol{v})\,\mathrm{d}\boldsymbol{v} = 1$. The operator

$$\mathrm{D}_\Omega = \frac{\partial}{\partial t} + \boldsymbol{v} \,.\, \nabla - \boldsymbol{v} \times \boldsymbol{\Omega} \,.\, \left(\frac{\partial}{\partial \boldsymbol{v}}\right) \equiv \mathrm{D} - \boldsymbol{v} \times \boldsymbol{\Omega} \,.\, \left(\frac{\partial}{\partial \boldsymbol{v}}\right)$$

where $\boldsymbol{\Omega} = e\boldsymbol{B}_0/mc$. The electric and magnetic fields are related to the potentials by the relations

$$\boldsymbol{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} \qquad \boldsymbol{B} = \nabla \times \boldsymbol{A}. \tag{30}$$

From (29) expressions for $\mathscr{L}_1$, $\mathscr{L}_2$ and $\mathscr{L}_3$ may be written as follows:

$$\mathscr{L}_1 = f_0(\boldsymbol{v})\left(m\boldsymbol{v} \,.\, (\mathrm{D}_\Omega \boldsymbol{r}) + e\phi^{(1)} - \frac{e\boldsymbol{v}}{c} \,.\, \{(\boldsymbol{r} \,.\, \nabla)\boldsymbol{A}_0 + \boldsymbol{A}^{(1)}\} - \frac{e}{c}\mathrm{D}_\Omega \boldsymbol{r} \,.\, \boldsymbol{A}_0\right)$$
$$+ \frac{\chi(\boldsymbol{v})}{4\pi}\{(\nabla \times \boldsymbol{A}_0) \,.\, (\nabla \times \boldsymbol{A}^{(1)})\}.$$

Integration by parts and use of the density conservation law, the zero-order Maxwell equations and zero-order Lorentz equation, gives $L_1 = 0$ as shown by Low (1958). Further

$$\mathscr{L}_2 = f_0(\boldsymbol{v})\left\{\frac{m}{2}(\mathrm{D}_\Omega \boldsymbol{r})^2 + e\left((\boldsymbol{r}\cdot\nabla)\phi^{(1)} - \frac{1}{c}\boldsymbol{v}\cdot(\boldsymbol{r}\cdot\nabla)\boldsymbol{A}^{(1)} - \frac{1}{2c}(\boldsymbol{r}\cdot\nabla)^2\boldsymbol{A}_0 - \frac{1}{c}\mathrm{D}_\Omega\boldsymbol{r}\cdot\{\boldsymbol{A}^{(1)} + (\boldsymbol{r}\cdot\nabla)\boldsymbol{A}_0\}\right)\right\} + \frac{\chi(\boldsymbol{v})}{8\pi}\left\{\left(-\nabla\phi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}\right)^2 - (\nabla\times\boldsymbol{A}^{(1)})^2\right\} \tag{31}$$

$$\mathscr{L}_3 = ef_0(\boldsymbol{v})\left\{\frac{1}{2}(\boldsymbol{r}\cdot\nabla)^2\phi^{(1)} - \frac{1}{c}\boldsymbol{v}\cdot\left(\frac{1}{2}(\boldsymbol{r}\cdot\nabla)^2\boldsymbol{A}^{(1)} + \frac{1}{6}(\boldsymbol{r}\cdot\nabla)^3\boldsymbol{A}_0\right) - \frac{1}{c}\mathrm{D}_\Omega\boldsymbol{r}\cdot\left((\boldsymbol{r}\cdot\nabla)\boldsymbol{A}^{(1)} + \frac{1}{2}(\boldsymbol{r}\cdot\nabla)^2\boldsymbol{A}_0\right)\right\}. \tag{32}$$

The subscripts 0 denote zero-order quantities and superscripts (1) denote first order quantities. The variation of $\mathscr{L}$ with respect to $\boldsymbol{r}$ yields the Lorentz equation

$$m\mathrm{D}_\Omega^2\boldsymbol{r} = -e\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B}\right) \tag{33}$$

whilst variations with respect to ϕ and $\boldsymbol{A}$ yield the Maxwell equations

$$\nabla\cdot\boldsymbol{E} = -4\pi e\int \mathrm{d}\boldsymbol{v}\, f$$

$$\nabla\times\boldsymbol{B} = \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} - 4\pi e\int \mathrm{d}\boldsymbol{v}\,\boldsymbol{v}f.$$

3. Application of the method

In this section we examine two examples of wave interactions which have been studied previously by conventional methods. The generation of longitudinal plasma waves by two high frequency electromagnetic waves in a warm field-free plasma, and the interaction of three electromagnetic waves in a cold magnetized plasma is examined.

In these examples, only spatial variation is considered, so that the coupled mode equations become

$$\begin{aligned} \frac{\mathrm{i}}{\omega_{1,2}}\boldsymbol{\Lambda}_{1,2}\cdot\frac{\partial}{\partial \boldsymbol{x}}\hat{a}_{1,2} &= \Gamma^*_{\mathrm{wc}}\hat{a}^*_{2,1}\hat{a}_3 \\ \frac{\mathrm{i}}{\omega_3}\boldsymbol{\Lambda}_3\cdot\frac{\partial}{\partial \boldsymbol{x}}\hat{a}_3 &= \Gamma_{\mathrm{wc}}\hat{a}_1\hat{a}_2. \end{aligned} \tag{34}$$

3.1. Generation of longitudinal plasma waves by transverse waves

Denoting the longitudinal mode by L and the transverse modes by T, the synchronism conditions are

$$\omega_3^{\mathrm{T}} - \omega_2^{\mathrm{T}} = \omega_1^{\mathrm{L}} \qquad \boldsymbol{k}_3^{\mathrm{T}} - \boldsymbol{k}_2^{\mathrm{T}} = \boldsymbol{k}_1^{\mathrm{L}}.$$

From (31) and (32) the lagrangian densities are given by

$$\mathscr{L}_2 = f_0(\boldsymbol{v})\left(\frac{m}{2}(\mathrm{D}\boldsymbol{r})^2 + e(\boldsymbol{r}\,.\,\nabla)\phi^{(1)} - \frac{e}{c}\boldsymbol{v}\,.\,\{(\boldsymbol{r}\,.\,\nabla)\boldsymbol{A}^{(1)}\} - \frac{e}{c}\mathrm{D}\boldsymbol{r}\,.\,\boldsymbol{A}^{(1)}\right)$$

$$+\frac{\chi(\boldsymbol{v})}{8\pi}\left\{\left(-\nabla\phi^{(1)} - \frac{1}{c}\frac{\partial \boldsymbol{A}^{(1)}}{\partial t}\right)^2 - (\nabla\times\boldsymbol{A}^{(1)})^2\right\} \tag{35}$$

$$\mathscr{L}_3 = ef_0(\boldsymbol{v})\left(\frac{1}{2}(\boldsymbol{r}\,.\,\nabla)^2\phi^{(1)} - \frac{1}{2c}\boldsymbol{v}\,.\,\{(\boldsymbol{r}\,.\,\nabla)^2\boldsymbol{A}^{(1)}\} - \frac{1}{c}\mathrm{D}\boldsymbol{r}\,.\,(\boldsymbol{r}\,.\,\nabla)\boldsymbol{A}^{(1)}\right) \tag{36}$$

where $f_0(\boldsymbol{v})$ is the equilibrium distribution function. The linear equations of motion, obtained from $\mathscr{L}_2$ are

$$\mathrm{D}^2\boldsymbol{r} = -\frac{e}{m}\left(\boldsymbol{E}^{(1)} + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B}^{(1)}\right)$$

which gives, using (10)

$$\boldsymbol{r} = \frac{e}{m}(\omega - \boldsymbol{k}\,.\,\boldsymbol{v})^{-2}\{\boldsymbol{E}^{(1)} + \omega^{-1}\boldsymbol{v}\times(\boldsymbol{k}\times\boldsymbol{E}^{(1)})\}. \tag{37}$$

For the high frequency modes $\omega_j \gg \boldsymbol{k}_j\,.\,\boldsymbol{v}$ ($j = 2, 3$) so that $\mathrm{D}\rightarrow\partial/\partial t$. Dropping the superscript (1) from the electric field, (37) gives

$$\boldsymbol{r}_1 = \frac{e}{m(\omega_1 - \boldsymbol{k}_1\,.\,\boldsymbol{v})^2}\boldsymbol{E}_1 \qquad \boldsymbol{r}_j = \frac{e}{m\omega_j^2}\boldsymbol{E}_j \qquad (j = 2, 3). \tag{38a, b}$$

For convenience, consider spatial variation in z alone and write

$$\boldsymbol{k} = (0, 0, k).$$

The integrated energy flux vector $\boldsymbol{P}^{\mathrm{s}}$ has only the component P_z^{s}, that is

$$P_z^{\mathrm{s}}(\boldsymbol{r}, \boldsymbol{A}, \phi, \boldsymbol{x}) = \int\left\{f_0(\boldsymbol{v})\left\{mv_z\left(\mathrm{D}\boldsymbol{r} - \frac{e}{c}\boldsymbol{A}\right)\,.\,\frac{\partial\boldsymbol{r}}{\partial t} - \frac{e}{c}r_z\boldsymbol{v}\,.\,\frac{\partial\boldsymbol{A}}{\partial t}\right\}\right.$$

$$\left. - \frac{\chi(\boldsymbol{v})}{4\pi}\frac{\partial\boldsymbol{A}}{\partial z}\,.\,\frac{\partial\boldsymbol{A}}{\partial t}\right\}\mathrm{d}\boldsymbol{v} \tag{39}$$

so that for transverse waves, since $\int v_z f_0(\boldsymbol{v})\,\mathrm{d}\boldsymbol{v} = 0$

$$P_z^{\mathrm{s}}(\boldsymbol{r}_j, \boldsymbol{A}_j, \boldsymbol{x}) = -\frac{1}{4\pi}\frac{\partial\boldsymbol{A}_j}{\partial z}\,.\,\frac{\partial\boldsymbol{A}_j}{\partial t}$$

that is

$$\overline{P_z^{\mathrm{s}}(\boldsymbol{r}_j, \boldsymbol{A}_j, \boldsymbol{x})} = \frac{1}{8\pi}\omega_j k_j\hat{\boldsymbol{A}}_j\,.\,\hat{\boldsymbol{A}}_j^* \qquad (j = 2, 3)$$

on space–time averaging. For the transverse modes, we take $\boldsymbol{E}_j = E_j(\cos\theta_j, \sin\theta_j, 0)$ ($j = 2, 3$) and choose to represent all the wave variables in terms of the individual electric field amplitudes $E_n (n = 1, 2, 3)$. Then

$$\overline{P_z^{\mathrm{s}}(\boldsymbol{r}_j, \boldsymbol{A}_j, \boldsymbol{x})} = \frac{c^2 k_j}{8\pi\omega_j}\hat{E}_j\hat{E}_j^* \qquad (j = 2, 3).$$

Thus from (27*b*)

$$\Lambda_j = \frac{c^2 k_j}{8\pi\omega_j} \qquad (j = 2, 3).$$

For the electrostatic mode, $\boldsymbol{A}_1 = \boldsymbol{0}$ and (39) becomes

$$P_z^{\mathrm{s}}(\boldsymbol{r}_1, \phi_1, \boldsymbol{x}) = \int f_0(\boldsymbol{v})\left(mv_z \mathrm{D}\boldsymbol{r}_1 \cdot \frac{\partial \boldsymbol{r}_1}{\partial t}\right) \mathrm{d}\boldsymbol{v}$$

that is

$$\overline{P_z^{\mathrm{s}}(\boldsymbol{r}_1, \phi_1, \boldsymbol{x})} = \int f_0(\boldsymbol{v})\left(mv_z \frac{\omega_1(\omega_1 - \boldsymbol{k}_1 \cdot \boldsymbol{v})}{2} \hat{\boldsymbol{r}}_1 \cdot \hat{\boldsymbol{r}}_1^*\right) \mathrm{d}\boldsymbol{v}.$$

Substituting for $\boldsymbol{r}_1$ from (38*b*) gives

$$\overline{P_z^{\mathrm{s}}} = \frac{e^2 \omega_1 \hat{E}_1 \hat{E}_1^*}{2m} \int f_0(\boldsymbol{v}) v_z (\omega_1 - k_1 v_z)^{-3}\, \mathrm{d}\boldsymbol{v}.$$

Assuming that $\omega_1 \gg \boldsymbol{k}_1 \cdot \boldsymbol{v}$, the integral equals $3k_1 n_0 v_{\mathrm{e}}^2/\omega_1^4$ where n_0 is the number density and $v_{\mathrm{e}} = (\kappa T/m)^{1/2}$. Hence

$$\Lambda_1 = \frac{3k_1 v_{\mathrm{e}}^2 \omega_{\mathrm{p}}^2}{8\pi\omega_1^3}$$

where ω_{p} is the plasma frequency. For the coupling coefficient Γ_{wc} ignoring the second term in (36) which vanishes on integration over velocity space and noting that

$$\boldsymbol{r} = \boldsymbol{r}_1 + \boldsymbol{r}_2 + \boldsymbol{r}_3 \qquad \phi = \phi_1 \qquad \boldsymbol{A} = \boldsymbol{A}_2 + \boldsymbol{A}_3$$

the space–time average of $\mathscr{L}_3$ is given by

$$\overline{\mathscr{L}_3} = \frac{ef_0(\boldsymbol{v})}{8c} \hat{r}_{1z}(\omega_2 k_3 \hat{\boldsymbol{r}}_2 \cdot \hat{\boldsymbol{A}}_3^* + \omega_3 k_2 \hat{\boldsymbol{A}}_2 \cdot \hat{\boldsymbol{r}}_3^*) + \mathrm{cc}$$

$$= \frac{\mathrm{i}e^3 k_1 \cos(\theta_2 - \theta_3)}{8m^2 \omega_1^2 \omega_2 \omega_3} f_0(\boldsymbol{v}) \hat{E}_1 \hat{E}_2 \hat{E}_3^* + \mathrm{cc}.$$

Integrating $\overline{\mathscr{L}_3}$ over velocity space leads to the coupling coefficient

$$\Gamma_{\mathrm{wc}} = \frac{\mathrm{i}e\omega_{\mathrm{p}}^2 k_1 \cos(\theta_2 - \theta_3)}{32\pi\omega_1^2 \omega_2 \omega_3 m} \tag{40}$$

and the coupled-mode equations become

$$\begin{aligned}
\frac{\partial \hat{E}_1}{\partial z} &= -\frac{e}{4m} \frac{\omega_1^2}{3v_{\mathrm{e}}^2 \omega_2 \omega_3} \cos(\theta_2 - \theta_3) \hat{E}_2^* \hat{E}_3 \\
\frac{\partial \hat{E}_2}{\partial z} &= -\frac{e}{4mc^2} \frac{k_1 \omega_{\mathrm{p}}^2 \omega_2}{k_2 \omega_1^2 \omega_3} \cos(\theta_2 - \theta_3) \hat{E}_1^* \hat{E}_3 \\
\frac{\partial \hat{E}_3}{\partial z} &= \frac{e}{4mc^2} \frac{k_1 \omega_{\mathrm{p}}^2 \omega_3}{k_3 \omega_1^2 \omega_2} \cos(\theta_2 - \theta_3) \hat{E}_1 \hat{E}_2.
\end{aligned} \tag{41}$$

These results then agree with those of Danilkin (1965). The important feature of this calculation has been the ease with which the coupling coefficients have been derived compared with a direct perturbation theory approach using the Vlasov–Maxwell equations.

3.2. Interaction between ordinary and extraordinary waves in a cold magnetized plasma

As a second example, illustrating the same point consider the interaction of three electromagnetic waves propagating at right angles to a uniform magnetic field $\boldsymbol{B}_0$ in a cold plasma. In particular let us examine the interaction between two ordinary modes (O), and an extraordinary mode (X) (Boyd and Sanderson 1969). In this case the synchronism conditions are

$$\omega_3^{\mathrm{O}} = \omega_1^{\mathrm{O}} + \omega_2^{\mathrm{X}} \qquad \boldsymbol{k}_3^{\mathrm{O}} = \boldsymbol{k}_1^{\mathrm{O}} + \boldsymbol{k}_2^{\mathrm{X}}.$$

The Lagrangians for a cold plasma can be written down from (35) and (36) by letting $f_0(\boldsymbol{v}) \equiv \delta(\boldsymbol{v})$. In this case $\mathrm{D} \to \partial/\partial t$ and

$$\mathscr{L}_2 = \frac{m}{2}\left(\frac{\partial \boldsymbol{r}}{\partial t}\right)^2 - \frac{e}{c}\left(\frac{\partial \boldsymbol{r}}{\partial t}\right) . \{\boldsymbol{A}^{(1)} + (\boldsymbol{r} . \nabla)\boldsymbol{A}_0\} + \frac{1}{8\pi}\left\{\frac{1}{c^2}\left(\frac{\partial \boldsymbol{A}^{(1)}}{\partial t}\right)^2 - (\nabla \times \boldsymbol{A}^{(1)})^2\right\} \tag{42}$$

$$\mathscr{L}_3 = -\frac{e}{c}\frac{\partial \boldsymbol{r}}{\partial t} . (\boldsymbol{r} . \nabla)\boldsymbol{A}^{(1)}. \tag{43}$$

Let $\boldsymbol{B}_0 = (0, 0, B_0)$ and $\boldsymbol{k} = (0, k, 0)$. The Euler–Lagrange equations for this case are

$$m\frac{\partial^2 \boldsymbol{r}}{\partial t^2} = -e\left(\boldsymbol{E}^{(1)} + \frac{1}{c}\frac{\partial \boldsymbol{r}}{\partial t} \times \boldsymbol{B}_0\right) \tag{44}$$

$$\nabla \times \boldsymbol{B}^{(1)} = \frac{1}{c}\frac{\partial \boldsymbol{E}^{(1)}}{\partial t} - \frac{4\pi n_0 e}{c}\frac{\partial \boldsymbol{r}}{\partial t}. \tag{45}$$

From (44)

$$\boldsymbol{r} = \frac{e}{m(\omega^2 - \Omega^2)}\begin{bmatrix} E_x - \mathrm{i}\dfrac{\Omega}{\omega}E_y \\ E_y + \mathrm{i}\dfrac{\Omega}{\omega}E_x \\ \left(\dfrac{\omega^2 - \Omega^2}{\omega^2}\right)E_z \end{bmatrix} \tag{46}$$

where all superscripts (1) have been dropped for convenience. Using $\nabla \times \boldsymbol{E} = -(1/c)(\partial \boldsymbol{B}/\partial t)$, (45) and (46) lead to the dispersion relation

$$(\omega^2 - \omega_{\mathrm{p}}^2 - c^2k^2)\left(\omega^2 - \omega_{\mathrm{p}}^2 - c^2k^2 + \frac{\omega_{\mathrm{p}}^2\Omega^2}{\Omega^2 + \omega_{\mathrm{p}}^2 - \omega^2}\right) = 0. \tag{47}$$

We now consider a general $\boldsymbol{k} = (k_x, k_y, 0)$ and for this problem it is convenient to introduce a new set of cartesian axes $\mathrm{O}x'y'z'$ where $\hat{\boldsymbol{z}}' = \hat{\boldsymbol{z}}$, $\hat{\boldsymbol{y}}' = \hat{\boldsymbol{k}}$ and $\hat{\boldsymbol{x}}' = \hat{\boldsymbol{e}} = \hat{\boldsymbol{k}} \times \hat{\boldsymbol{z}}$. All variables in the subsequent calculation are then referred to the common reference frame $Ox'y'z'$. In this frame of reference

$$\nabla \times A = \frac{\partial}{\partial y'}(\hat{e}A_{z'} - \hat{z}A_{x'}).$$

We now split r and A into their wave components so that

$$r = r_1 + r_2 + r_3$$

$$A = A_1 + A_2 + A_3$$

as in the general theory. Subscripts 1, 3 refer to the ordinary mode and 2 to the extraordinary mode. The energy flux vector now has one component $P_{y'}$ given by

$$P_{y'}(r, A, x) = \frac{\partial \mathscr{L}_2}{\partial(\partial_{y'} r)} \cdot \frac{\partial r}{\partial t} + \frac{\partial \mathscr{L}_2}{\partial(\partial_{y'} A)} \cdot \frac{\partial A}{\partial t}$$

which becomes, using (42)

$$P_{y'} = -\frac{1}{4\pi}\left(\frac{\partial A_{x'}}{\partial y'} \frac{\partial A_{x'}}{\partial t} + \frac{\partial A_{z'}}{\partial y'} \frac{\partial A_{z'}}{\partial t}\right)$$

that is

$$\overline{P_{y'}} = \frac{\omega_n k_n}{8\pi}(\hat{A}_{nx'}\hat{A}^*_{nx'} + \hat{A}_{nz'}\hat{A}^*_{nz'}). \tag{48}$$

From (46), we have for the ordinary modes

$$r_j = \frac{eE_j}{m\omega_j^2}\hat{z} \qquad (j = 1, 3) \tag{49}$$

and for the extraordinary mode

$$r_2 = \frac{e}{m(\omega_2^2 - \Omega^2)}\left(E_{2x'} - \mathrm{i}\frac{\Omega}{\omega_2}E_{2y'}, E_{2y'} + \mathrm{i}\frac{\Omega}{\omega_2}E_{2x'}, 0\right). \tag{50}$$

Since every wave parameter has to be expressed in terms of one parameter, we express $E_{2x'}$ in terms of $E_{2y'}$ using

$$-\frac{\mathrm{i}\omega_p^2\omega_2\Omega}{\omega_2^2 - \Omega^2}E_{2x'} + \left(\omega_2^2 - \frac{\omega_2^2\omega_p^2}{\omega_2^2 - \Omega^2}\right)E_{2y'} = 0$$

that is

$$E_2 = -\mathrm{i}aE_{2y'}\hat{e}_2 + E_{2y'}\hat{k}_2 \tag{51}$$

with $a = \mathrm{i}E_{2x'}/E_{2y'} = \omega_2(\omega_2^2 - \omega_{UH}^2)/\omega_p^2\Omega$ where $\omega_{UH} = (\omega_p^2 + \Omega^2)^{1/2}$ is the upper hybrid frequency. For the ordinary modes $E_j = E_j\hat{z}$ and for consistency we must write $E_2 = E_2\hat{u}$ where $\hat{u}$ is a unit vector, that is

$$E_2 = \frac{E_2}{(1 + a^2)^{1/2}}(-\mathrm{i}a\hat{e}_2 + \hat{k}_2) \tag{52}$$

and substituting (52) into (50) gives

$$r_2 = \frac{-\mathrm{i}e\omega_2 E_2}{m\omega_p^2\Omega(1 + a^2)^{1/2}}\left(\left(1 - \frac{\omega_p^2}{\omega_2^2}\right), \frac{\mathrm{i}\Omega}{\omega_2}, 0\right).$$

We are now in a position to calculate the various coupling coefficients. For the ordinary modes, from (48)

$$\overline{P_{y'}(E_j)} = \frac{c^2 k_j}{8\pi\omega_j}\hat{E}_j\hat{E}_j^* \qquad (j = 1, 3)$$

so that

$$\Lambda_j = \frac{c^2 \boldsymbol{k}_j}{8\pi\omega_j} \qquad (j = 1, 3).$$

For the extraordinary mode $E_{2z'} = 0$, $\hat{E}_{2x'}\hat{E}_{2x'}^* = a^2/(1+a^2)\hat{E}_2\hat{E}_2^*$ giving

$$\overline{P_{y'}(E_2)} = \frac{c^2 k_2 a^2}{8\pi\omega_2(1+a^2)}\hat{E}_2\hat{E}_2^*$$

and

$$\Lambda_2 = \frac{c^2 \boldsymbol{k}_2 a^2}{8\pi\omega_2(1+a^2)}.$$

Finally, the coupling coefficient Γ_{wc} is derived from

$$\mathscr{L}_3 = -\frac{e}{c} r_{y'} \frac{\partial \boldsymbol{r}}{\partial t} \cdot \frac{\partial \boldsymbol{A}}{\partial y'}$$

so that

$$\overline{\mathscr{L}_3} = \frac{e\hat{r}_{2y'}}{8c}(\omega_1 k_3 \hat{\boldsymbol{r}}_1 \cdot \hat{\boldsymbol{A}}_3^* + \omega_3 k_1 \hat{\boldsymbol{r}}_3^* \cdot \hat{\boldsymbol{A}}_1) + \mathrm{cc}$$

$$= \frac{e\hat{r}_{2y'}}{8c}\left(\frac{\mathrm{i}ec}{m\omega_1\omega_3}(k_3 - k_1)\right)\hat{E}_1\hat{E}_3^* + \mathrm{cc}.$$

Hence

$$\Gamma_{\mathrm{wc}} = \frac{\mathrm{i}e\omega_{\mathrm{p}}^2 k_2 \Omega}{32\pi m\beta\omega_1\omega_2\omega_3}$$

where

$$\beta^2 = \left(\frac{\Omega^2}{\omega_2^2}\right)\omega_{\mathrm{p}}^4 + (\omega_2^2 - \omega_{\mathrm{UH}}^2)^2.$$

Defining

$$V(\omega_1|\omega_2|\omega_3) = \frac{e\omega_{\mathrm{p}}^2 k_2 \omega_3 \Omega}{4mc^2\omega_1\omega_2\beta}$$

the coupled-mode equations become

$$(\boldsymbol{k}_1 \cdot \nabla)\hat{E}_1 = \left(\frac{\omega_1}{\omega_3}\right)^2 V(\omega_1|\omega_2|\omega_3)\hat{E}_2\hat{E}_3^*$$

$$(\boldsymbol{k}_2 \cdot \nabla)\hat{E}_2 = \left(\frac{\omega_2}{\omega_3}\right)^2\left(\frac{1+a^2}{a^2}\right) V(\omega_1|\omega_2|\omega_3)\hat{E}_1\hat{E}_3^* \qquad (53)$$

$$(\boldsymbol{k}_3 \cdot \nabla)\hat{E}_3 = V(\omega_1|\omega_2|\omega_3)\hat{E}_1\hat{E}_2.$$

To compare this result with Etievant *et al* (1968) we interchange 1 with 3 and take the complex conjugate of the first two equations in (53). Then noting that

$$\boldsymbol{k}_i \cdot \frac{\partial E_i}{\partial \boldsymbol{x}} = \frac{k_i^2}{(\boldsymbol{k}_i \cdot \boldsymbol{n})} \frac{\partial E_i}{\partial s}$$

where $\boldsymbol{x} = \boldsymbol{n}s$, the results are identical.

4. Discussion

The examples in § 3 illustrate the value of the lagrangian approach in plasma physics to which Low first drew attention. The comparative ease with which coupled mode equations are obtained means that this method offers a great advantage over the standard approach starting from the Vlasov–Maxwell equations. Indeed the saving of effort is even more evident in the case of three-wave interactions in a warm, magnetized plasma (considered in part II). The lagrangian approach also makes possible a discussion of four-wave interactions in cases where three-plasmon interactions are forbidden by the frequency and wavevector conservation relations. One important example in this category is the wave–wave interaction involving ion acoustic waves.

The basic steps in the theory presented in § 2 may be summarized here:

(i) starting with a lagrangian density $\mathscr{L}(q^i, \boldsymbol{x}, \boldsymbol{v})$ the q^i is separated into a component describing the equilibrium state, q_0^i, and one, η^i, describing the perturbation of this state due to waves and their interactions. $\mathscr{L}$ is then expanded in powers of the perturbation.

(ii) The perturbation is then split into individual wave components and separate, time-dependent wave Lagrangians $\mathscr{L}(n)$, are constructed.

(iii) Multiple length and time scales are used to describe the wave coupling, the key step in the calculation being the application of a space–time averaging to the equations governing the energy transfer of each wave participating in the interaction. This leads to equations describing rates of transfer of action for each individual wave in terms of space–time averaged generic wave energy densities, $\overline{\mathscr{H}_2(\eta_n^i, \boldsymbol{x}, \boldsymbol{v})}$, fluxes $\overline{\mathscr{P}_2^s(\eta_n^i, \boldsymbol{x}, \boldsymbol{v})}$ and coupling energy $\overline{\mathscr{L}_3(\eta_1^i + \eta_2^i + \eta_3^i, \boldsymbol{x}, \boldsymbol{v})}$.

(iv) Use of the linear equations of motion, derived from $\mathscr{L}_2$ allows each η_n^i to be expressed in terms of a single wave parameter, and the above quantities may then be written in terms of this parameter.

(v) Coupled-mode equations follow with coupling coefficients given directly in terms of averaged energy densities and fluxes.

Two key requirements have to be met before the lagrangian approach may be used in dealing with the class of weakly nonlinear problems described. The equations of motion must be derivable from a variational principle. Secondly, in describing the evolution of the plasma one must be able to assign multiple length and time scales so that space–time averaging over distances and periods long compared with k_n^{-1}, ω_n^{-1} may be carried out.

In part II the method is applied to describe wave interactions in warm, magnetized plasmas (Boyd and Turner 1971), together with the nonlinear interaction between positive and negative energy waves leading, under certain circumstances, to explosive instabilities.

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